TROPICAL INVARIANTS FROM THE SECONDARY FAN

ERIC KATZ

Abstract. In this paper, we consider weighted counts of tropical plane curves of particular combinatorial type through a certain number of generic points. We give a criterion, derived from tropical intersection theory on the secondary fan, for a weighted count to give a number invariant of the position of the points. By computing a certain intersection multiplicity, we show how Mikhalkin's approach to computing Gromov-Witten invariants fits into our approach. This begins to address a question raised by Dickenstein, Feichtner, and Sturmfels. We also give a geometric interpretation of the numbers we produce involving Chow quotients, and provide a counterexample showing that the tropical Severi variety is not always supported on the secondary fan.

1. Introduction

Tropical enumeration of curves was introduced by Mikhalkin [14], and has been studied by Itenberg, Kharlamov, and Shustin [7], Gathmann and Markwig [5], and Nishinou and Siebert [15] among others.

Mikhalkin's approach is to count tropical curves of certain combinatorial types through a number of points in general position. Each curve is weighted by an integer that has to do with the combinatorial type of its dual subdivision. This count turns out to be independent of the position of the points and is equal to Gromov-Witten invariants. There is a similar tropical count for Welschinger invariants and for relative Gromov-Witten invariants [5]. We ask the following question: *which other weights of tropical curves give a number that is independent of the position of the points.* We call such numbers *invariants.*

We search for such weights using a beautiful combinatorial object called the secondary fan. Let P be a closed lattice polygon in $\mathbb{Z}^2$, and $\mathcal{A} = P \cap \mathbb{Z}^2$. Associated to $\mathcal{A}$ is the secondary polytope in $\mathbb{R}^{\mathcal{A}}$. Its normal fan, Σ, called the secondary fan, parameterizes regular subdivisions of P which are a super-set of combinatorial types of tropical curves in $\mathbb{R}^2$. We look at weighted m-dimensional fans supported on the secondary fan. These are unions of m-dimensional cones in Σ weighted by rational numbers and are called *tropical cycles.* By intersecting these cycles with hyperplanes that impose point conditions, we may produce intersection numbers that are weighted counts of tropical curves through points in the plane. There is a balancing condition on the rational weights that will ensure that the intersection numbers are independent of the choice of generic point conditions and are therefore invariants. Because certain cones in the secondary fan are enumeratively irrelevant, we may relax the balancing condition and get even more invariants.

The invariants that we produce are new and have not, to our knowledge, been studied before. We hope that these invariants can shed light on tropical curve

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enumeration, give insight to more general degenerations in relative Gromov-Witten theory, and possibly provide new algorithms.

Next, we ask how we can recover Mikhalkin's algorithm for curve counting. This algorithm counts *nodal* tropical curves, that is, curves whose dual subdivision consists of triangles and parallelograms that pass through an appropriate number of points. These curves are weighted with multiplicity

$$\prod_{\Delta}(2\mathrm{area}(\Delta))$$

where the product is over triangles in the dual subdivision. This weighted count computes Gromov-Witten invariants of the toric variety $X_{\mathcal{A}}$ associated to $\mathcal{A}$.

We find the intersection multiplicity coming from the cones in the secondary fan corresponding to nodal tropical curves by applying the following theorem, the main result of this paper.

Theorem 1.1. *Let C be an effective, nodal cone of dimension m in the secondary fan in $T\mathbb{P}^{|\mathcal{A}|-1}$. Let $H'_{p_1}, H'_{p_2}, \ldots, H'_{p_m}$ be point-condition hyperplanes corresponding to points in general position. Then any point $q \in C \cap H'_{p_1} \cap \cdots \cap H'_{p_m}$ occurs with multiplicity equal to*

$$\frac{\prod_{\Delta}(2\mathrm{area}(\Delta))\prod_{\square}\mathrm{area}(\square)}{\prod_{E\in I\mathcal{E}} l(E)}$$

the product of twice the areas of the triangles times the product of the area of the parallelograms divided by the product of the lattice lengths of the internal edges.

The terms *effective* and *point-condition hyperplane* will be defined below. We also reformulate the intersection-theoretic statement in the language of the the intersection theory of tropical fans developed by Gathmann, Kerber, and Markwig [4]. One may take the fan supported on the secondary fan where the m-dimensional cone corresponding to an effective, nodal subdivision is weighted by

$$\frac{\prod_{E\in I\mathcal{E}} l(E)}{\prod_{\square}\mathrm{area}(\square)}.$$

Intersecting such a fan with point conditions will compute Gromov-Witten invariants by the above theorem. This comparison of the intersection theory of the secondary fan with Gromov-Witten invariants begins to address a question raised by Dickenstein, Feichtner, and Sturmfels in [2].

There is a geometric interpretation of the invariants we produce. It involves viewing the toric variety associated to the secondary fan, the Chow quotient $(\mathbb{P}^{|\mathcal{A}|})//(\mathbb{C}^*)^2$ as a moduli space of toric surfaces in which we find curves. The tropical fans supported on the secondary fan correspond to Chow cohomology classes on the Chow quotient. These classes force the target space to degenerate. This is similar to the situation in relative Gromov-Witten theory and we hope that our invariants are related to the not-yet-developed theory of relative Gromov-Witten invariants of a toric surface relative its non-smooth toric boundary.

When we began this project, we expected the fan that computes Gromov-Witten invariants to be the tropicalization of the Severi variety V_δ, the closure of the locus of curves in $\mathbb{P}^{|\mathcal{A}|-1}$ that have δ nodes. When the weights above turned out not to always be integers, we became aware that this was certainly not the case. For completeness, we provide an example of a cone in the tropicalization of the Severi variety that is not a cone in the secondary fan.

The structure of the paper is as follows. In section 2, we review basic definitions in tropical geometry. We marshall several results in tropical intersection theory from [4] and [3] in section 3. Section 4 summarize the necessary properties of the secondary fan. In section 5, we provide conditions for the intersection numbers of weighted fans supported on the secondary fan to give invariants. Section 6 gives several examples of invariants. Section 7 relates the multiplicity that occurs in the computation of invariants to a more tractable lattice index which is used in section 8 to prove Theorem 1.1. Section 9 gives the geometric interpretation and section 10, the counterexample to the tropical Severi variety being supported on the secondary fan.

This paper is a substantial revision of the preprint "The Tropical Degree of Cones in the Secondary Fan." The main addition in the revision of the use of the tropical intersection theory of fans from [4] to justify the intersection theory computations and (we hope) improve readability.

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2. Background

We adopt the point of view of [16]. The field of Puiseux series will be denoted by $\mathbb{K} = \mathbb{C}\{\{t\}\}$. Let

$$\mathrm{ord} : \mathbb{K} \setminus \{0\} \to \mathbb{Q}$$

denote the valuation defined by

$$\mathrm{ord}(c_1 t^{q_1} + c_2 t^{q_2} + \dots) = q_1$$

where $c_i \neq 0$, $q_i \in \mathbb{Q}$, and $q_1 < q_2 < \dots$. A tropical variety will be defined as the closure of the image of a variety $W \subset \mathbb{K}^n$ under the order map $\mathrm{Trop} : (\mathbb{K} \setminus \{0\})^n \to \mathbb{Q}^n$.

A hypersurface in $\mathbb{K}^2$ can be defined by an equation

$$\sum_{(i,j)} a_{ij} x^i y^j = 0$$

where $a_{ij} \in \mathbb{K}$. By Kapranov's theorem, this hypersurface's corresponding tropical variety is the locus of points $(\overline{x}, \overline{y}) \in \mathbb{R}^2$ where the minimum

$$\min(\mathrm{ord}(a_{ij}) + i\overline{x} + j\overline{y})$$

is achieved twice.

Observe that if $a_{ij} \in \mathbb{C}$, $\mathrm{ord}(a_{ij}) = 0$ and $\mathrm{Trop}(V(f))$ is a fan.

3. Intersection Theory

We will use tropical intersection theory as announced in [13]. Because the proofs of these results have not yet been proven, we will cite specific results from [4] and [3]. It suffices to use only tropical fans for our purposes rather than more general tropical varieties.

Let us recall some definitions from [4]. We will use weights in a semigroup $G \subset \mathbb{R}$ where [4] used positive integer ones, but all proofs hold as before. Let V be a vector space with a distinguished full-rank lattice Λ. For σ, an integral cone in V, let V_σ

be the linear span of σ, and Λ_σ denote $V_\sigma \cap \Lambda$, a full-rank lattice in V_σ. For $X \subset V$ an integral fan, let $X^{(n)}$ denote the n-dimensional cones in X.

Definition 3.1. *Let Λ, Λ' be lattices. Let X be a fan in $V = \Lambda \otimes \mathbb{R}$ and let Y be a fan in $V' = \Lambda' \otimes \mathbb{R}$. A morphism $f : X \to Y$ is a $\mathbb{Z}$-linear map from $|X| \subset V$ to $|Y| \subset V'$.*

Definition 3.2. *For $G \subset \mathbb{R}$ a semigroup, a* G-weighted fan *in V is a pair (X, ω_X) where X is an integral fan of some pure dimension n in V and $\omega_X : X^{(n)} \to G$ is a function. We call $\omega_X(\sigma)$ the* weight *of the cone $\sigma \in X^{(n)}$.*

In the case of a G-weighted fan (X, ω_X), we may use $|X|$ to refer to the underlying point set of the fan.

Definition 3.3. *Let $\tau < \sigma$ be cones in V with $\dim \tau = \dim \sigma - 1$. Let $u_{\sigma/\tau} \in \Lambda/\Lambda_\tau$ be the unique generator of $\Lambda_\sigma/\Lambda_\tau$ that lies in the positive span of σ in V_σ/V_τ.*

Definition 3.4. *A G-weighted* tropical fan *in V is a G-weighted fan (X, ω_X) in V such that for all $\tau \in X^{(\dim X - 1)}$, the balancing condition*

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot u_{\sigma/\tau} = 0 \in V/V_\tau$$

holds.

Definition 3.5. *A $\mathbb{Z}_{>0}$-weighted tropical fan $X \subset V$ is* irreducible *if there is no $\mathbb{Z}_{>0}$-weighted tropical fan Y of the same dimension in V with $|Y| \subsetneq |X|$.*

Observe that $\mathbb{R}^n$ with fan structure consisting of a single cone is irreducible.

We will find the following generalization of the balancing condition useful.

Definition 3.6. *Let $f : V \to V'$ be a linear map. Let (X, ω_X) be a G-weighted fan in V. (X, ω_X) is said to be* weakly balanced with respect to f *if for all $\tau \in X^{(\dim X - 1)}$ such that f is injective on V_τ,*

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot u_{\sigma/\tau} \in \mathrm{Ker}(f)/V_\tau.$$

There is an alternative formulation of weak balancing as follows. A top-dimensional cone σ is said to be f-ineffective if f is not injective on σ. A weighted fan is weakly balanced with respect to f if for every codimension-one cone τ on which f is injective, we may (after adding some ineffective cones containing τ), modify the weights on some ineffective cones containing τ so that the balancing condition is satisfied at τ.

The image of a tropical fan X under a morphism $f : X \to X'$ is defined in [4]. The proof of Proposition 2.25 of [4] proves the following slight generalization:

Proposition 3.7. *Let X be an n-dimensional G-weighted fan in $V = \Lambda \otimes \mathbb{R}$, X' an arbitrary fan in $V' = \Lambda \otimes \mathbb{R}$, and $f : X \to X'$ a morphism. If f is weakly balanced with respect to f, then $f(X)$ is an n-dimensional tropical fan as well.*

This gives the following Corollary which is a modification of Corollary 2.26 of [4].

Corollary 3.8. *Let X and X' be fans of the same dimension n in $V = \Lambda \otimes \mathbb{R}$ and $V' = \Lambda' \otimes \mathbb{R}$, respectively. Let $f : X \to X'$ be a morphism. Assume X is weakly balanced with respect to f and X' is irreducible. Then there is a fan X'_0 in V' of smaller dimension with $|X'_0| \subset |X'|$ such that*

(a) *each point* $q \in |X'| \setminus |X'_0|$ *lies in the interior of a cone* $\sigma'_q \in X'$ *of dimension* n*;*
(b) *each point* $p \in f^{-1}(|X'| \setminus |X'_0|)$ *lies in the interior of a of a cone* $\sigma_p \in X$ *of dimension* n*;*
(c) *for* $q \in |X'| \setminus |X'_0|$ *the sum*

$$\sum_{p \in |X|: f(p)=q} \mathrm{mult}_p(f)$$

does not depend on q *where the* multiplicity $\mathrm{mult}_p(f)$ *of* f *at* p *is defined to be*

$$\mathrm{mult}_p(f) = \frac{\omega_X(\sigma_p)}{\omega'_X(\sigma'_q)} \cdot |\Lambda'_{\sigma'_q}/f(\Lambda_{\sigma_p})|$$

We need the definition of tropical intersections and the pullback of a tropical fan. They are both defined in [13] but the proof of their well-definedness has not yet been published. We therefore must use an equivalent definition coming from the intersection theory of toric varieties [3]. Mikhalkin's definition specializes to the one we use in the case of tropical fans. The connection between tropical varieties and Chow cohomology of toric varieties comes from an observation of Bernd Sturmfels and is elaborated in [10].

Definition 3.9. *Let* X *be a* G*-weighted tropical fan of dimension* k *and* Δ *is a complete fan so that each cone of* X *is a union of cones in* Δ*. Let* $\Delta^{(l)}$ *be the codimension* l *cones in* Δ*, and* $K(G)$ *be the smallest group in* $\mathbb{R}$ *containing* G*. The* Minkowski weight *of codimension* $n-k$ *associated to* X *is the map*

$$c : \Delta^{(n-k)} \to K(G)$$

defined by $c(\sigma) = \omega_X(\sigma')$ *if* σ *is contained in a cone* $\sigma' \in X^{(k)}$*.*

The balancing condition translates exactly into the Minkowski weight condition. The group of Minkowski weights of codimension k is isomorphic to $A^k(X(\Delta)) \otimes K(G)$.

Definition 3.10. *Let* X *and* Y *be tropical fans of dimension* k *and* l *in* V*. Let* Δ *be a complete fan so that each cone of* X *and* Y *is a union of cones in* Δ*. Consider* X *and* Y *as Chow cohomology classes,* $c \in A^{n-k}(X(\Delta)) \otimes K(G)$ *and* $d \in A^{n-l}(X(\Delta)) \otimes K(G)$*. Then* $X \cdot Y$*, the tropical intersection of* X *and* Y *corresponds to* $c \cup d \in A^{2n-k-l}(X(\Delta)) \otimes K(G)$*.*

We may unwind the above definition since $c \cup d$ is defined as a Minkowski weight. For γ a cone in Δ of dimension $k+l-n$,

$$(c \cup d)(\gamma) = \sum_{(\sigma,\tau)} m^{\gamma}_{\sigma,\tau} \cdot c(\sigma) \cdot d(\tau)$$

where the sum is over cones σ, τ with $\dim(\sigma) = k$, $\dim(\tau) = l$, $\gamma \subset \sigma$, and $\gamma \subset \tau$. The structure constant $m^{\gamma}_{\sigma,\tau}$ can be defined as follows: let $v \in \Lambda$ be chosen generic in the sense of [3] then

$$m^{\gamma}_{\sigma,\tau} = \begin{cases} |\Lambda/(\Lambda_\sigma + \Lambda_\tau)| & \text{if } \sigma \cap (\tau + v) \neq \emptyset \\ 0 & \text{else} \end{cases}$$

We will not use the specifics of the genericity condition on v except to note that there are finitely many hyperplanes in Λ such that every vector in the complement of their union is generic.

Definition 3.11. *The* degree *of a zero dimensional tropical cycle is the sum of the weights of its points.*

The pullback of a tropical cycle is defined in section 4.3 of [13]. It specializes as follows.

Definition 3.12. *Let $X \subset \mathbb{R}^{n_1+n_2}$ and $X' \subset \mathbb{R}^{n_2}$ be tropical fans of dimension k, k'. Let $f : X \to X'$ be a morphism of tropical fans induced by the projection $\mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_2}$. We define f^*X' to be the tropical intersection $(\mathbb{R}^{n_1} \times X') \cdot X$*

Proposition 3.13. *For f as above, f^*X' is well-defined.*

Proof. Let Δ, Δ' be complete fans so that each k-dimensional cell of X, X' is a union of cells of dimension at most k, k' in Δ, Δ', respectively. By refining Δ, we may suppose that f is a morphism of fans. The tropical fans X, X' are Chow cohomology classes, $c \in A^{n_1+n_2-k}(X(\Delta)) \otimes K(G), c' \in A^{n_1-k'}(X(\Delta')) \otimes K(G)$, seen as Minkowski weights. By applying Propositions 3.7 and 4.1 of [3], we see the pullback f^*X' is the Minkowski weight corresponding to $f^*c' \cup c$. This is well-defined. $\square$

We elaborate on the definition of the pullback fan in the above proof. The Minkowski weights corresponding to X, X' are function $c : \Delta \to K(G)$, $c' : \Delta' \to K(G)$. Then f^*c is the following function on Δ: for $\sigma \in \Delta$ with σ', the smallest cone of Δ' containing $f(\sigma)$,

$$(f^*c')(\sigma) = \begin{cases} c'(\sigma') & \text{if} \dim(\sigma) = n_2 + k' \\ 0 & \text{else} \end{cases}.$$

For γ a cone in Δ of dimension $k + k' - n_1$,

$$(f^*c' \cup c)(\gamma) = \sum_{(\sigma,\tau)} m^{\gamma}_{\sigma,\tau} \cdot f^*c'(\sigma) \cdot c(\tau)$$

where the sum is over cones σ, τ with $\dim(\sigma) = n_2 + k'$, $\dim(\tau) = k$, $\gamma \subset \sigma$, and $\gamma \subset \tau$.

The lattice index in the definition of $m^{\gamma}_{\sigma,\tau}$ can be written in another form using the following lemma from [10]:

Lemma 3.14. *Let Λ_1 and Λ_2 be saturated lattices in $\mathbb{Z}^n$ of complementary rank so that $\Lambda_1 + \Lambda_2$ has rank n. Then*

$$|\mathbb{Z}^n/(\Lambda_1 + \Lambda_2)| = \left|(\mathbb{Z}^n)^\vee/(\Lambda_1^\perp + \Lambda_2^\perp)\right|$$

where

$$\Lambda_i^\perp = \ker((\mathbb{Z}^n)^\vee \to \Lambda_i^\vee).$$

Here one notes that when computing $|\Lambda/(\Lambda_\sigma + \Lambda_\tau)|$ that Λ_γ is a saturated sublattice of all the lattices involved. Therefore, the lattice index is equal to

$$|(\Lambda/\Lambda_\gamma)/((\Lambda_\sigma/\Lambda_\gamma) + (\Lambda_\tau/\Lambda_\gamma))| = \left|\Lambda_\gamma^\perp/(\Lambda_\sigma^\perp + \Lambda_\tau^\perp)\right|.$$

It will be useful to write down the weights on an intersection of codimension 1 cones. We begin with the following standard result

Lemma 3.15. *Let $f(x) = \min_{u \in U}(x \cdot u)$ be a tropical polynomial where x are coordinates on $\Lambda \otimes \mathbb{R}$ and $U \subset \Lambda^\vee$ is a set of exponents. Let X be the codimension 1 tropical fan cut out by f. Let σ be a top-dimensional cone in X and $U^\sigma \subset U$ be the set of exponents so that σ is defined by*

$$x \cdot u = x \cdot u'$$

for $u, u' \in U^\sigma$,

$$x \cdot u < x \cdot v$$

for $u \in U^\sigma, v \in U \setminus U^\sigma$. Suppose U^σ has two elements u, u'. Let Λ_{U^σ} be the lattice

$$\Lambda_{U^\sigma} = \mathbb{Z} \cdot (u - u').$$

Then the weight on σ is $\left|\Lambda_\sigma^\perp / \Lambda_{U^\sigma}\right|$.

Lemma 3.16. *Let $X_1, \dots, X_m$ be codimension-one tropical fans in V. Let Y be the tropical intersection*

$$Y = X_1 \cdot \dots \cdot X_m.$$

Suppose that their point-wise intersection is an m-dimensional fan such that any open top-dimensional cone γ lies in the relative interior of top-dimensional cones $\sigma_1, \dots, \sigma_m$ of $X_1, \dots, X_m$, respectively. Then $|Y| = \cap_{i=1}^n |X_i|$ and the weight on γ in Y is given by

$$\omega_Y(\gamma) = \omega_{X_1}(\sigma_1) \dots \omega_{X_m}(\sigma_m) \left|\Lambda_\gamma^\perp / (\Lambda_{\sigma_1}^\perp + \dots + \Lambda_{\sigma_m}^\perp)\right|.$$

Proof. The statement about underlying point sets follows from definitions. We only need to prove the statement about weights. Let $\tau_k = \cap_{i=1}^k \sigma_i$ and $Z_k = \cdot_{i=1}^k X_i$. Since the linear span of $\tau_{k-1} \cup \sigma_k$ is all of V, we may choose v generic so that $\sigma_k \cap (\tau_{k-1} + v) = \sigma_k \cap \tau_{k-1}$. Then,

$$\omega_{Z_k}(\tau_k) = \omega_{X_i}(\sigma_i) \cdot \omega_{Z_{k-1}}(\tau_{k-1}) \cdot \left|\Lambda / (\Lambda_{\tau_{k-1}} + \Lambda_{\sigma_k})\right|$$

But $\Lambda_{\tau_{k-1}} \cap \Lambda_{\sigma_k} = \Lambda_{\tau_k}$. So the lattice index is

$$\left|(\Lambda/\Lambda_{\tau_k}) / ((\Lambda_{\tau_{k-1}}/\Lambda_{\tau_k}) + (\Lambda_{\sigma_k}/\Lambda_{\tau_k}))\right| = \left|\Lambda_{\tau_k}^\perp / (\Lambda_{\tau_{k-1}}^\perp + \Lambda_{\sigma_k}^\perp)\right|.$$

Therefore,,

$$\begin{aligned}
\omega_Y(\gamma) &= \omega_{Z_m}(\tau_m) \\
&= \omega_{X_1}(\sigma_1) \dots \omega_{X_m}(\sigma_m) \left|\Lambda_{\tau_m}^\perp / (\Lambda_{\tau_{m-1}}^\perp + \Lambda_{\sigma_m}^\perp)\right| \cdot \dots \cdot \left|\Lambda_{\tau_2}^\perp / (\Lambda_{\tau_1}^\perp + \Lambda_{\sigma_2}^\perp)\right| \\
&= \omega_{X_1}(\sigma_1) \dots \omega_{X_m}(\sigma_m) \left|\Lambda_\gamma^\perp / (\Lambda_{\sigma_1}^\perp + \dots + \Lambda_{\sigma_m}^\perp)\right|.
\end{aligned}$$

□

4. The Secondary Fan

We state some definitions involving the secondary fan [6]. Let P be a closed lattice polygon in $\mathbb{Z}^2$. Let $\mathcal{A} = P \cap \mathbb{Z}^2$. We can study tropical hypersurfaces in $\mathbb{R}^2$ supported on $\mathcal{A}$, that is, ones cut out by the tropicalization of the equation

$$\sum_{(i,j) \in \mathcal{A}} a_{ij} x^i y^j = 0.$$

Define $\psi \in \mathbb{R}^{\mathcal{A}}$ by

$$\psi(i,j) = a_{i,j}.$$

Let the upper hull $UH(\psi)$ of ψ be the convex hull of the subset of $\mathbb{R}^3$ given by

$$S = \{(i,j,a)|(i,j) \in \mathcal{A}, a \geq \psi(i,j)\}.$$

The lower faces of $UH(\psi)$ project to P giving a regular subdivision. The tropical variety is dual to the regular subdivision. Define the lower convexity $LC(\psi)$ of ψ to be the function $LC(\psi) : P \to \mathbb{R}$ whose graph is the lower faces of $UH(\psi)$. From ψ, we get a marked subdivision of P whose faces, P_k are the domains of linearity of $LC(\psi)$ and whose vertices are

$$\mathcal{A}_k = \{(i,j) \in P_k|\psi(i,j) = LC(\psi)(i,j).$$

The lattice points (i,j) for which $\psi(i,j) > LC(\psi)(i,j)$ are called *missing lattice points*. We call the data of $\{(P_k, \mathcal{A}_k)\}$ the *marked combinatorial type*.

Given $\psi \in \mathbb{R}^{\mathcal{A}}$, we may form the minimal cone of the secondary fan containing ψ following [6]. Let $C(\psi)$ be the cone in $\mathbb{R}^{\mathcal{A}}$ for which $\chi \in C(\psi)$ if and only if

(1) The subdivision associated to χ is the subdivision associated to ψ or a coarsening of it
(2) $\chi(i,j) = LC(\chi)(i,j)$ for all (i,j) for which $\psi(i,j) = LC(\psi)(i,j)$.

Therefore, the relative interior of cones correspond to marked combinatorial type.

These cones fit together to form a fan called the *secondary fan*. Every cone in the secondary fan contains the 3-dimensional linearity space, K consisting of all functions of the form

$$\psi(i,j) = ai + bj + c$$

for $a, b, c \in \mathbb{R}$.

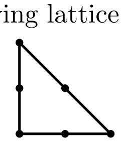
The line L given by the constant functions can be projected out. Let $T\mathbb{P}^{|\mathcal{A}|-1}$ denote $\mathbb{R}^{\mathcal{A}}/L$. We may treat the secondary fan as a fan Σ in $T\mathbb{P}^{|\mathcal{A}|-1}$. The lattice in $T\mathbb{P}^{\mathcal{A}-1}$ is $\mathbb{Z}^{\mathcal{A}}/\mathbb{Z}$.

Definition 4.1. *A function $\psi : \mathcal{A} \to \mathbb{R}$ is said to be* effective *if $\psi = LC(\psi)|_{\mathcal{A}}$ or equivalently if ψ is convex.*

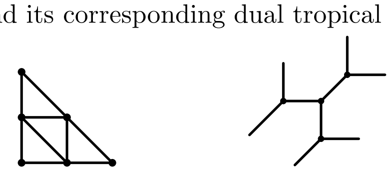
Note that if ψ is effective then every function in $C(\psi)$ is effective. Such cones are called *effective*. They are equivalently characterized as having no missing lattice points.

The secondary fan is the normal fan of a polytope in $\mathbb{R}^{\mathcal{A}}$ called the secondary polytope. As such, the positive codimension cones of the secondary fan with appropriate weights form a tropical fan.

Example 4.2. Let $\mathcal{A}$ be the following lattice points.



Here is a subdivision and its corresponding dual tropical curve.



This tropical curve corresponds to a point in the relative interior of a 5-dimensional cone in the secondary fan in $T\mathbb{P}^{|\mathcal{A}|-1}$.

Definition 4.3. *A cone in the secondary fan is said to be* nodal *if its corresponding subdivision consists only of triangles and parallelograms.*

We have the following straightforward dimension count.

Lemma 4.4. *The dimension of a cone $C \subset \Sigma \subset T\mathbb{P}^{|\mathcal{A}|-1}$ in the secondary fan obeys the inequality*

$$m \geq \#MLP + \#\mathcal{V} - \sum_{F \in \mathcal{F}} (E(F) - 3) - 1 = \#MLP + 3\#\mathcal{F} - 2\#I\mathcal{E} + \#I\mathcal{V} - 1$$

where $\#MLP$ is the number of missing lattice points, $\mathcal{F}$, $I\mathcal{E}$, $\mathcal{V}$, $I\mathcal{V}$ are the faces, internal edges, vertices and internal vertices of the regular subdivision, and $E(F)$ is the number of edges of a face. For an effective nodal subdivision, this becomes an equality (by [14]*) and simplifies to*

$$m = V - \#\square - 1 = \frac{1}{2}\#\Delta + \frac{1}{2}\#E\mathcal{E}$$

where $\#\square, \#\Delta, \#E\mathcal{E}$ are the numbers of parallelograms, triangles, external edges respectively.

Definition 4.5. *For $(i,j) \in \mathcal{A}$, let $\phi_{ij} \in (\mathbb{R}^{\mathcal{A}})^{\vee}$ be given by*

$$\phi_{ij}(\psi) = \psi(i,j)$$

for $\psi \in \mathbb{R}^{\mathcal{A}}$.

5. Tropical Invariants

5.1. **Tropical Fans Supported on the Secondary Fan.** In this section, we describe how to obtain invariants from tropical fans supported on the secondary fan.

Definition 5.1. *An m-dimensional* tropical fan supported on Σ *is a $\mathbb{Q}$-weighted tropical fan (X, ω_X) such that $|X|$ is a union of cones in $\Sigma \subset T\mathbb{P}^{|\mathcal{A}|-1}$.*

Definition 5.2. *If $p = (\overline{x}, \overline{y}) \in \mathbb{Q}^2$, the* point condition hyperplane *corresponding to p is the tropical variety in $T\mathbb{P}^{|\mathcal{A}|-1}$ cut out by the tropicalization of*

$$\sum_{(i,j) \in \mathcal{A}} a_{ij} (t^{\overline{x}})^i (t^{\overline{y}})^j = 0.$$

We start with a provisional definition of the invariant associated to a fan supported on the secondary fan.

"Definition" 5.3. *For an m-dimensional tropical fan X, supported on Σ, the invariant associated to X is the degree of the intersection*

$$\deg(X \cdot H_{(\overline{x}_1, \overline{y}_1)} \cdot H_{(\overline{x}_2, \overline{y}_2)} \cdot \cdots \cdot H_{(\overline{x}_m, \overline{y}_m)})$$

for "generically chosen" $(\overline{x}_i, \overline{y}_i) \in \mathbb{Q}^2$.

Because cones in the secondary fan correspond to marked combinatorial types of curves in $\mathbb{R}^2$, the invariant is a weighted count of curves of particular marked combinatorial types passing through m generic points. The fact that such an invariant is well-defined follows from Mikhalkin's intersection theory, but we formulate a definition using the intersection theory of tropical cones due to Gathmann, Kerber, and Markwig. [4]

Definition 5.4. *For a positive integer l, the* incidence correspondence*, V_l is the tropical fan in $T\mathbb{P}^{|\mathcal{A}|-1} \times (\mathbb{R}^2)^l$ given as tropical intersection of $H_1, \ldots, H_l$ where H_k is the codimension one tropical fan cut out by*

$$\sum_{(i,j)\in\mathcal{A}} a_{ij} x_k^i y_k^i = 0$$

for $k = 1, \ldots, l$ where (x_k, y_k) are coordinates on the $(\mathbb{R}^2)$'s.

Observe that V_l has tropical morphisms

$$\begin{aligned} \pi_{\mathcal{A}} &: V_l \to T\mathbb{P}^{|\mathcal{A}|-1} \\ \pi_m &: V_l \to \mathbb{R}^{2l} \end{aligned}$$

Note that because that each defining equation for V_l is a tropical equation involving (x_k, y_k) for different values of k, V_l satisfies the hypotheses of Lemma 3.16, and top dimensional cones in V_l are of codimension l.

Definition 5.5. *Let X be an m-dimensional tropical fan supported on Σ. The pullback to V_m, $\pi_{\mathcal{A}}^* X$ is a $2m$-dimensional tropical fan. Let $\mathbb{R}^{2m}$ be considered as a fan with one $2m$-dimensional cone, weighted by 1. The* invariant *associated to X is the degree of the morphism $\pi_m : \pi_{\mathcal{A}}^* X \to \mathbb{R}^{2m}$.*

Definition 5.6. *For an m-dimensional cone $C \subset \Sigma$, let $V_m(C)$ be the polyhedral complex in $T\mathbb{P}^{|\mathcal{A}|-1} \times \mathbb{R}^{2m}$ given by $\pi_{\mathcal{A}}^{-1}(C) \cap V_m$.*

5.2. **Effectively Balanced Tropical Fans.** Now, there are some m-dimensional cones $C \subset \Sigma$ such that if γ is a $2m$-dimensional cone in V_l with $\pi_{\mathcal{A}}(\gamma) = C$ then π_m is not injective on γ. These cones cannot possibly contribute to the invariant. This allows us to relax the balancing condition on X. We need to assemble some results to show that this is possible. Observe that a point in $V_m(C)$ is a tropical curve with marked combinatorial type corresponding to C with m (not necessarily distinct) marked points lying on it. Top-dimensional cones in $V_m(C)$ are $2m$-dimensional.

Lemma 5.7. *If C is not an effective cone, then the projection $\pi_m : V_m(C) \to \mathbb{R}^{2m}$ is not injective on any cone in $V_m(C)$.*

Proof. If the cone is not effective then there exists $(i_0, j_0) \in \mathcal{A}$ such that for every $\psi \in C$, $\psi(i_0, j_0) \geq LC(\psi)(i_0, j_0)$. Then $a_{i_0 j_0}$ can be varied without changing the tropical variety,

$$\{ \sum_{(i,j)\in\mathcal{A}} a_{ij} x^i y^j = 0 \}.$$

This will therefore not affect the image under π_m. □

This lemma makes the ineffective cones, in a particular sense, enumeratively irrelevant.

Definition 5.8. *An m-dimensional G-weighted fan X supported on $N(\Sigma)$ is said to be* effectively balanced *if the following holds for all $\tau \in X^{(\dim X - 1)}$: let $\nu_1, \ldots, \nu_l$ be the ineffective cones containing τ, then*

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot u_{\sigma/\tau} \in \mathrm{Span}(u_{\nu_i/\tau}).$$

Lemma 5.9. *An effectively balanced fan X on Σ pulls back by $\pi_{\mathcal{A}}$ to a fan supported on V_m that is weakly balanced with respect to π_m.*

Proof. We use the techniques of [3]. Let Δ be a fan in $T\mathbb{P}^{|\mathcal{A}|-1} \times \mathbb{R}^{2m}$ so that V_m is a union of cones of Δ and set $\Delta' = \Sigma$. Therefore, V_m is represented by some class $c \in A^m(X(\Delta))$. Let τ be a $(2m-1)$-dimensional cone in V_m mapping to an $(m-1)$-dimensional cone $\tau' \in \Delta'$. We must show how to define $\pi_{\mathcal{A}}^* X$ and that it is balanced at τ.

Consider the orbit closures $V(\tau), V(\tau')$. There is a commutative diagram

$$\begin{tikzcd} V(\tau) \arrow[r, hook, "i"] \arrow[d, "\pi"'] & X(\Delta) \arrow[d, "\pi_{\mathcal{A}}"] \\ V(\tau') \arrow[r, hook, "i'"'] & X(\Delta') \end{tikzcd}$$

Choose weights on the ineffective cones containing τ' so that the balancing condition holds at τ'. This gives a cohomology class $c' \in A^*(V(\tau'))$. The pullback weights on the cones is $\pi^* c' \cup i^* c$ and is therefore balanced. The weights on the effective cones is independent of the choices that were made. Therefore the pullback fan is weakly balanced with respect to $\pi_{\mathcal{A}}$ at every τ. $\square$

Theorem 5.10. *An effectively balanced m-dimensional $\mathbb{Q}$-weighted fan on Σ gives an invariant.*

6. Examples of Invariants

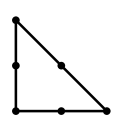
Example 6.1. The union of top-dimensional cones of the secondary polytope, each weighted by 1 is a tropical fan. The associated invariant is, of course, 1.

Example 6.2. The union of facets of the secondary fan weighted by the lattice lengths of the corresponding edge of the secondary polytope is a tropical fan. The corresponding sum of cones is the tropicalization of the principal $\mathcal{A}$-determinant (if it is not 1) by chapter 10 of [6].

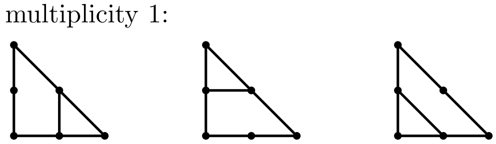
Example 6.3. Any union of rays of the secondary fan in $\mathbb{R}^{|\mathcal{A}|}/K$ that satisfies the balancing condition at $0 \in \mathbb{R}^{|\mathcal{A}|}/K$ is a tropical fan.

Example 6.4. The linearity space K, weighted by 1 is a tropical fan. The corresponding subdivision is an undivided lattice polygon.

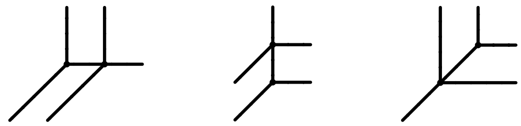
Example 6.5. (The TV Antenna) Let $\mathcal{A}$ be the following lattice points



Consider the union of rays in $\mathbb{R}^{\mathcal{A}}/K$ corresponding to the following subdivisions, each weighted with multiplicity 1:



This is effectively balanced and the corresponding tropical invariant is a weighted count of tropical curves of the following form through three generic points:



The value of the invariant can be computed to be 6.

Example 6.6. As stated in the introduction, we may take the fan consisting of m-dimensional effective nodal cones where each cone is weighted by

$$\frac{\prod_{E \in I\mathcal{E}} l(E)}{\prod_{\square} \operatorname{area}(\square)}.$$

This fan gives a well-defined degree, equal to the corresponding Gromov-Witten invariant by Mikhalkin's correspondence theorem [14]. We do not know if this fan is effectively balanced. We can also consider related formal sums that give the relative tropical invariants of Gathmann and Markwig [5] and the Welschinger invariants [14]. One should note that these weights may not be integral. This corresponds to the fact that the tropicalization of the Severi variety is not always supported on cones in the secondary fan. See section 10 for a cone in the tropicalization of the Severi variety that is not any cone in the secondary fan.

The paper of Billera and Sturmfels [1] on fiber polytopes is a very rich source of invariant formal sums. This paper gives many Minkowski sum decompositions of the fiber polytope. Because the normal fan of a Minkowski sum is the refinement of the normal fans of the summands, a tropical fan supported on the normal fan of a Minkowski summand induces a tropical fan supported on the secondary fan and, therefore, an invariant. In particular the invariants coming from the boundary fibers $\Sigma(\Delta^{|\mathcal{A}|}, \partial_i \operatorname{Conv}(\mathcal{A}))$ (see section 3 of [1]) may be related to relative Gromov-Witten invariants.

7. Degree and Lattice Index

For a m-dimensional cone C in $\Sigma \subset T\mathbb{P}^{|\mathcal{A}|-1}$, consider $\pi_m : V_m(C) \to \mathbb{R}^{2m}$. Let γ be a $2m$-dimensional cone in $V_m(C)$. Let $p \in \gamma^\circ$, $q = \pi_m(q) \in \mathbb{R}^{2m}$. To facilitate computations, it will be necessary to relate the multiplicity at p in Corollary 3.8 to a more computable lattice index.

We must first gain some understanding of the equations defining γ. A $2m$-dimensional cone γ in $V_m(C)$ is cut out from $\pi_{\mathcal{A}}^{-1}(C)$ by linear equations of the form

$$a_{i_k j_k} + i_k x_k + j_k y_k = a_{i'_k j'_k} + i'_k x_k + j'_k y_k$$

for $k = 1, \ldots, m$ and $(i_k, j_k), (i'_k, j'_k) \in \mathcal{A}$ together with some inequalities. We will find which choices of $(i_k, j_k), (i'_k, j'_k)$ can occur.

Lemma 7.1. *Let C be a m-dimensional cone in $\Sigma \subset T\mathbb{P}^{|\mathcal{A}|-1}$. Let γ be a $2m$-dimensional cone in $V_m(C)$. Then γ is cut out from $\pi_{\mathcal{A}}^{-1}(C)$ by some inequalities and equations of the form*

$$a_{i_k j_k} + i_k x_k + j_k y_k = a_{i'_k j'_k} + i'_k x_k + j'_k y_k$$

for $(i_k, j_k), (i'_k, j'_k)$ lying in an edge of the subdivision corresponding to C.

Proof. The equations that cut out γ from $\pi_{\mathcal{A}}^{-1}(C)$ must each impose exactly one condition. Let $(\psi, (\overline{x}_k, \overline{y}_k)) \in \gamma^\circ \subset T\mathbb{P}^{|\mathcal{A}|-1} \times \mathbb{R}^{2m}$. Suppose that $(i_k, j_k), (i'_k, j'_k)$

do not lie along the edge of the dual subdivision. Temporarily replace ψ by $\psi - i_k\overline{x}_k - j_k\overline{y}_k - a_{i_kj_k}$ so that $\psi(i_k, j_k) = \psi(i'_k, j'_k) = 0$ while $\psi(i, j) \geq 0$.

Let S be the segment between (i_1, j_1) and (i_2, j_2). First note that if S crosses the interior of a face of the regular subdivision then $\psi = 0$ along that face. This imposes two conditions on the a_{ij}'s. Therefore S must lie along edges of the regular subdivision. If S contains a vertex of the regular subdivision in its interior, then ψ of that vertex must be 0. Therefore the slope of the edges on either side of that vertex along S must be zero. This imposes two conditions. □

Let $\sigma = \pi_{\mathcal{A}}^{-1}(C)$. To find the pullback of X, we must ask which cones $\tau \subset V_m$ intersect the cone $\sigma + v$ for $v \in (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z}) \times \mathbb{Z}^{2m}$ generic. Let χ_{ij} be unit basis elements on $\mathbb{Z}^{\mathcal{A}}$. We may consider them as a spanning set of $\mathbb{Z}^{\mathcal{A}}/\mathbb{Z}$.

Lemma 7.2. *Let $\epsilon : \mathcal{A} \to \mathbb{Z}$ be a strictly convex function. Let $v = \sum_{i,j} \epsilon(i, j)\chi_{ij}$ viewed as an element of $(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z}) \times \mathbb{Z}^{2m}$. Let C be an m-dimensional cone in Σ, $\sigma = \pi_{\mathcal{A}}^{-1}(C)$, and γ, a $2m$-dimensional cone in $V_m(C)$. Then a codimension m cone $\tau \in V_m$ with $\tau \supset \gamma$ has $\sigma \cap (\tau - v) \neq \emptyset$ only if the cone τ is defined by some inequalities together with equalities*

$$a_{i_kj_k} + i_kx_k + j_ky_k = a_{i'_kj'_k} + i'_kx_k + j'_ky_k$$

where $(i_k, j_k), (i'_k, j'_k)$ are the endpoints of edges in the dual subdivision for $k = 1, \ldots, m$.

Proof. A cone $\tau - v$ is cut out by equations of the form:

$$a_{i_kj_k} + i_kx_k + j_ky_k - \epsilon(i_k, j_k) = a_{i'_kj'_k} + i'_kx_k + j'_ky_k - \epsilon(i'_k, j'_k) \leq a_{ij} + ix_k + jy_k - \epsilon(i, j)$$

where $(i_k, j_k), (i'_k, j'_k)$ are points on the same edge, E in the subdivision. Pick a point $(a_{ij}, (\overline{x}_k, \overline{y}_k))$ in the relative interior of $\sigma \cap (\tau - v)$.

Let $a_0, \ldots, a_e$ and $\epsilon_0, \ldots, \epsilon_e$ be the values of a_{ij} and $\epsilon(i, j)$ along the lattice points lying on the edge. Then the above equality/inequality becomes

$$a_{l_k} + l_k\overline{z} - \epsilon_{l_k} + d = a_{l'_k} + l'_k\overline{z} - \epsilon_{l'_k} + d \leq a_l + l\overline{z} - \epsilon_l + d$$

where $\overline{z}$ is a particular linear combination of $\overline{x}, \overline{y}$ depending on the slope of E, d is a particular constant, and l indexes lattice points on E. The secondary fan condition gives

$$a_l = b + mc$$

for $b, m \in \mathbb{R}$. Therefore, we have

$$l_k(m + \overline{z}) - \epsilon_{l_k} = l'_k(m + \overline{z}) - \epsilon_{l'_k} \leq l(m + \overline{z}) - \epsilon_l.$$

In other words, if $c = l_k(m + \overline{z}) - \epsilon_{l_k}$ and $h(l) = l(m + \overline{z}) - c$, then

$$\epsilon_l \leq h(l)$$

with

$$\epsilon_{l_k} = h(l_k), \quad \epsilon_{l'_l} = h(l'_k).$$

In view of the convexity of ϵ, l_k and l'_k must be the endpoints of E. □

We can choose ϵ as above so that v is generic in the sense of [3]. It follows that for a given $2m$-dimensional cone $\gamma \subset V_m(C)$ there is a unique $\tau \subset V_m$ such that $m^{\gamma}_{\sigma,\tau} \neq 0$ for $\sigma = \pi_{\mathcal{A}}^{-1}(C)$.

Definition 7.3. *Let $\Lambda_C^\perp \subset (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\perp$ be the lattice of integer forms that are 0 on $C \subset T\mathbb{P}^{|\mathcal{A}|-1}$. Let the point-condition lattice, $\Lambda_{PC,\tau} \subset (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\perp \oplus (\mathbb{Z}^{2m})^\perp$ be the lattice of integer forms generated by the equations from the above lemma that are 0 on τ.*

Observe that each of the equations form the above lemma give a saturated one-dimensional lattice in $(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\perp \oplus (\mathbb{Z}^{2m})^\perp$. Therefore, by Lemma 3.16 we have

Lemma 7.4. *The weight on a cone τ as above in V_m is*

$$\omega_{V_m}(\tau) = \left|\mathbb{Z}_\tau^\perp/\Lambda_{PC,\tau}\right|.$$

We may denote $\Lambda_{PC,\tau}$ by $\Lambda_{PC,\gamma}$ when there is no confusion. Let $\pi_{\mathcal{A}} : (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee \times (\mathbb{Z}^{2m})^\vee \to (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee$ be the projection. Then $\pi_{\mathcal{A}}(\Lambda_{PC,\tau})$ is a lattice in $(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee$ spanned by

$$\phi_{i_k j_k} - \phi_{i'_k j'_k}$$

as above.

Lemma 7.5. *Let X be an effectively balanced m-dimensional fan supported on Σ. Let C be a cone in Σ. For γ, a $2m$-dimensional cone in $V_m(C)$, the weight on γ in $\pi_{\mathcal{A}}^* X$ is given by*

$$\omega_{\pi_{\mathcal{A}}^* X}(\gamma) = \omega_X(C)\left|\Lambda_\gamma^\perp/(\pi_{\mathcal{A}}^*\Lambda_C^\perp + \Lambda_{PC,\tau})\right|.$$

Proof. Let $\sigma = \pi_{\mathcal{A}}^{-1}(C)$. Then $\Lambda_\sigma^\perp = \pi_{\mathcal{A}}^*\Lambda_C^\perp$. By the definition of intersection product from [3] and Lemma 3.14,

$$\begin{aligned} \omega_{\pi_{\mathcal{A}}^* X}(\gamma) &= \omega_X(C)\cdot\omega_{V_m}(\tau)\cdot\left|\Lambda_\gamma^\perp/(\Lambda_\sigma^\perp + \Lambda_\tau^\perp)\right| \\ &= \omega_X(C)\cdot\left|\Lambda_\gamma^\perp/(\pi_{\mathcal{A}}^*\Lambda_C^\perp + \Lambda_{PC,\tau})\right|. \end{aligned}$$

□

Lemma 7.6. *Let C be an m-dimensional cone in Σ. Let γ be a $2m$-dimensional cone in $V_m(C)$. Let $p \in \gamma^\circ$ and $q = \pi_m(p)$. The multiplicity,*

$$\mathrm{mult}_p(\pi_m) = \omega_{\pi_{\mathcal{A}}^*(X)}(\gamma)\left|\mathbb{Z}^{2m}/\pi_m(\Lambda_\gamma)\right|$$

is equal to

$$\omega_X(C)\left|(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee/(\Lambda_C^\perp + \pi_{\mathcal{A}}(\Lambda_{PC,\gamma}))\right|.$$

Proof. Let $\sigma = \pi_{\mathcal{A}}^{-1}(C)$ and τ be the unique cone in $V_m(C)$ with $m^\gamma_{\sigma,\tau} \neq 0$. Observe that

$$\left|\mathbb{Z}^{2m}/\pi_m(\Lambda_\gamma)\right| = \left|((\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})\oplus\mathbb{Z}^{2m})/(\Lambda_\gamma + (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z}))\right|$$

which, by Lemma 3.14, is equal to

$$\begin{aligned} \left|((\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee\oplus(\mathbb{Z}^{2m})^\vee)/(\Lambda_\gamma^\perp + (\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\perp)\right| &= \left|((\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee\oplus(\mathbb{Z}^{2m})^\vee)/(\Lambda_\gamma^\perp + (\mathbb{Z}^{2m})^\vee)\right| \\ &= \left|(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee/\pi_{\mathcal{A}}(\Lambda_\gamma^\perp)\right|. \end{aligned}$$

Combining this equality with the one from the lemma above, we get that $\mathrm{mult}_p(\pi_m)$ is equal to

$$\omega_X(C)\cdot\left|\Lambda_\gamma^\perp/(\pi_{\mathcal{A}}^*(\Lambda_C^\perp) + \Lambda_{PC,\tau})\right|\cdot\left|(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee/\pi_{\mathcal{A}}(\Lambda_\gamma^\perp)\right|.$$

But for this to be non-zero $\pi_{\mathcal{A}}$ must be injective on $\Lambda_\gamma^\perp$. Therefore, this quantity must equal

$$\omega_X(C)\cdot\left|(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee/(\Lambda_C^\perp + \pi_{\mathcal{A}}(\Lambda_{PC,\tau}))\right|.$$

□

Let us fix an effective cone C in the secondary fan. Consider a cone γ in $V_m(C)$ as above. Let us color the edges in the regular subdivision corresponding to the edges $(i_k, j_k), (i'_k, j'_k)$, obtaining a subgraph of the 1-skeleton of the regular subdivision. If the lattice index in the above lemma is to be well-defined, the graph can have no circuits and is called the *point condition forest*.

Since it is often inconvenient to work with the projectivization $T\mathbb{P}^{|\mathcal{A}|-1}$, we may break the scaling symmetry.

Definition 7.7. *Let $(I, J) \in \mathcal{A}$ be a vertex of the regular subdivision. Define the* enriched point condition lattice $\Lambda'_{PC,\gamma}$ *by* $\Lambda'_{PC,\gamma} = \pi_{\mathcal{A}}(\Lambda_{PC,\gamma}) + \phi_{IJ}$.

It is straightforward to see

$$\left|(\mathbb{Z}^{\mathcal{A}}/\mathbb{Z})^\vee/(\Lambda_C^\perp + \pi_{\mathcal{A}}(\Lambda_{PC,\gamma}))\right| = \left|(\mathbb{Z}^{\mathcal{A}})^\vee/(\Lambda_C^\perp + \Lambda'_{PC,\gamma})\right|.$$

8. Lattice Index for Nodal Effective Cones

In this section, we prove Theorem 1.1 which will compute $\left|(\mathbb{Z}^{\mathcal{A}})^\vee/(\Lambda_C^\perp + \Lambda'_{PC,\gamma})\right|$ if C is an effective nodal cone.

Let us first reformulate the theorem in the language of multiplicities.

Theorem 8.1. *Let X be an m-dimensional effectively balanced cone supported on $\Sigma \subset T\mathbb{P}^{|\mathcal{A}|-1}$. Let C be an effective, nodal top-dimensional cone in X, weighted by 1. C corresponds to a marked subdivision of P. Let γ be a $2m$-dimensional cone in $V_m(C)$. Then, for $p \in \gamma^\circ$, the multiplicity contributing to the degree in Corollary 3.8 is*

$$\mathrm{mult}_p(\pi_m) = \frac{\prod_\Delta(2\mathrm{area}(\Delta))\prod_\Box \mathrm{area}(\Box)}{\prod_{E\in I\mathcal{E}} l(E),}$$

the product of twice the areas of the triangles times the product of the area of the parallelograms divided by the product of the lattice lengths of the internal edge in the subdivision.

One should note that in the nodal case, the multiplicity is an invariant of the cone. For more general cones in the secondary fan, the lattice index may depend on the edges in the regular subdivision on which the point condition occurs.

We will cite a number of lemmas which will be proved below. The strategy of the proof is to break up the lattice index computation into a lattice index computation for each cell in the subdivision. We will make use of the following standard lemma.

Lemma 8.2. *Give an exact sequence of free $\mathbb{Z}$-modules*

$$0 \longrightarrow \mathbb{Z}^{m_0} \longrightarrow \mathbb{Z}^{m_1} \longrightarrow \cdots \longrightarrow \mathbb{Z}^{m_r} \longrightarrow 0$$

with an exact sequence of full rank sublattices

$$0 \longrightarrow \Lambda_0 \longrightarrow \Lambda_1 \longrightarrow \cdots \longrightarrow \Lambda_r \longrightarrow 0$$

then

$$\prod_{i=0}^r |\mathbb{Z}^{m_i}/\Lambda_i|^{(-1)^i} = 1$$

Proof. By standard homological algebra,

$$0 \longrightarrow \mathbb{Z}^{m_0}/\Lambda_0 \longrightarrow \mathbb{Z}^{m_1}/\Lambda_1 \longrightarrow \cdots \longrightarrow \mathbb{Z}^{m_r}/\Lambda_r \longrightarrow 0$$

is exact. ∎

For C a cone in Σ, let V be the minimal subspace of $\mathbb{R}^{\mathcal{A}}$ containing C, $\Lambda_C^\perp$ be the lattice in $(\mathbb{Z}^{\mathcal{A}})^\vee$ consisting of all functions ϕ satisfying $\phi(V)=0$.

Let $\mathcal{F}$,$I\mathcal{E}$,$I\mathcal{V}$ be the set of faces, internal edges (those with a face on either side), and internal vertices (those surrounded by faces) of the regular subdivision. For each $F\in\mathcal{F}$ or $E\in\mathcal{E}$, let $\mathcal{A}_F,\mathcal{A}_E$ be the lattice points in F and E, respectively.

Definition 8.3. *For $F\in\mathcal{F}$, let $V_F\subseteq\mathbb{R}^{\mathcal{A}_F}$ be the set of all functions $a:\mathcal{A}_F\to\mathbb{R}$ such that the set $\{(v,a(v))|v\in F\}$ lies in a plane. For $E\in I\mathcal{E}$ let $V_E\subseteq\mathbb{R}^{\mathcal{A}_E}$ be the set of all functions $a:\mathcal{A}_E\to\mathbb{R}$ so that $\{(v,a(v))|v\in F\}$ lies on a line. Let $V_v=\mathbb{R}^{\{v\}}$.*

Definition 8.4. *For $?=F,E,v$, let $V_?^\perp$ be the linear subspace of $(\mathbb{R}^{\mathcal{A}_?})^\vee$ consisting of all functions ϕ such that $\phi(V_?)=0$. Let $\Lambda_?^\perp$ be the lattice $V_?^\perp\cap(\mathbb{Z}^{\mathcal{A}_?})^\vee$.*

Definition 8.5. *For $F\in\mathcal{F}$, a triangle, we define $\Lambda'_{PC,F}$ to be the lattice in $(\mathbb{Z}^{\mathcal{A}_F})^\vee$ spanned by*

$$\phi_{v_0}-\phi_{v_1},\phi_{v_1}-\phi_{v_2},\phi_{v_0}$$

or alternatively

$$\phi_{v_0},\phi_{v_1},\phi_{v_2}$$

Definition 8.6. *For $F\in\mathcal{F}$, a parallelogram with vertices v_0,v_1,v_2,v_3 occurring counterclockwise, we define $\Lambda'_{PC,F}$ to be the lattice in $(\mathbb{Z}^{\mathcal{A}_F})^\vee$ spanned by*

$$\phi_{v_0}-\phi_{v_1},\phi_{v_0}-\phi_{v_2},\phi_{v_0}-\phi_{v_1}+\phi_{v_2}-\phi_{v_3},\phi_{v_0}$$

or alternatively

$$\phi_{v_0},\phi_{v_1},\phi_{v_2},\phi_{v_3}$$

Notice that the third term in the basis is not a point condition but is rather borrowed from $\Lambda_F^\perp$. This is for convenience.

Definition 8.7. *For $E\in I\mathcal{E}$, define $\Lambda'_{PC,E}$ to be the lattice in $(\mathbb{Z}^{\mathcal{A}_E})^\vee$ spanned by*

$$\phi_{v_0}-\phi_{v_1},\phi_{v_0}$$

or alternatively

$$\phi_{v_0},\phi_{v_1}$$

where v_0,v_1 are the endpoints of E.

Definition 8.8. *For $v\in I\mathcal{E}$, let $\Lambda'_{PC,v}=(\mathbb{Z}^{\{v\}})^\vee$.*

The lattices are tied together by the exact sequence of ambient lattices and the exact sequence of $L+PC$.

Lemma 8.9. *The following sequence is exact*

$$0\longrightarrow\bigoplus_{v\in I\mathcal{V}}(\mathbb{Z}^{\{v\}})^\vee\longrightarrow\bigoplus_{E\in I\mathcal{E}}(\mathbb{Z}^{\mathcal{A}_E})^\vee\longrightarrow\bigoplus_{F\in\mathcal{F}}(\mathbb{Z}^{\mathcal{A}_F})^\vee\longrightarrow(\mathbb{Z}^{\mathcal{A}})^\vee\longrightarrow 0$$

Proposition 8.10. *The following sequence is exact*

$$0\longrightarrow\bigoplus_{v\in I\mathcal{V}}\Lambda_v^\perp+\Lambda'_{PC,v}\xrightarrow{d_0}\bigoplus_{E\in I\mathcal{E}}\Lambda_E^\perp+\Lambda'_{PC,E}$$

$$\xrightarrow{d_1}\bigoplus_{F\in\mathcal{F}}\Lambda_F^\perp+\Lambda'_{PC,F}\xrightarrow{d_2}\Lambda_C^\perp+\Lambda'_{PC,\gamma}\longrightarrow 0$$

We can apply Lemma 8.2 to compute $|(\mathbb{Z}^{\mathcal{A}})^\vee/(\Lambda_C^\perp + \Lambda'_{PC,\gamma})|$ once we know the lattice indices of the other terms in the sequence.

Lemma 8.11. *For F a triangle, the lattice index of $\Lambda_F^\perp + \Lambda'_{PC,F}$ in $(\mathbb{Z}^{\mathcal{A}_F})^\vee$ is equal to* $2\mathrm{area}(F)$.

Lemma 8.12. *For F a parallelogram, the lattice index of $\Lambda_F^\perp + \Lambda_{PC,F}$ in $(\mathbb{Z}^{\mathcal{A}_F})^\vee$ is equal to* $\mathrm{area}(F)$.

Lemma 8.13. *For E an edge of length e, the lattice index of $\Lambda_E^\perp + \Lambda_{PC,E}$ in $(\mathbb{Z}^{\mathcal{A}_E})^\vee$ is equal to e.*

There is also the following trivial lemma:

Lemma 8.14. *For v a vertex, the lattice index of $\Lambda_v^\perp + \Lambda_{PC,v}$ in $(\mathbb{Z}^{\mathcal{A}_v})^\vee$ is equal to* 1.

Now, by assembling the above results, we obtain Theorem 1.1.

8.1. **Inclusion/Exclusion Exact Sequence.** In the subsection, we prove Lemma 8.9 which associates a four-term exact sequence to an effective regular subdivision of a lattice polygon. We call this the inclusion/exclusion exact sequence.

The signs in the exact sequence are only defined after we pick orientations which we do below.

Definition 8.15. *An FE-flag is a pair (F, E), $F \in \mathcal{F}$, $E \in \mathcal{E}$ such that $E \subset F$.*

Definition 8.16. *A system of orientations is a set of FE-flags such that each internal edge occurs exactly once and no other edges occur. The system of orientations can be visualized as an arrow across each internal edge.*

We will construct the following exact sequence:

$$0 \longrightarrow \mathbb{R}^{\mathcal{A}} \xrightarrow{\ i\ } \bigoplus_{F\in\mathcal{F}} \mathbb{R}^{\mathcal{A}_F} \xrightarrow{\ d_2\ } \bigoplus_{E\in I\mathcal{E}} \mathbb{R}^{\mathcal{A}_E} \xrightarrow{\ d_1\ } \bigoplus_{v\in I\mathcal{V}} \mathbb{R}^{\{v\}} \longrightarrow 0.$$

The map i is the direct sums of maps induced by the canonical inclusion

$$i_F : \mathcal{A}_F \hookrightarrow \mathcal{A}.$$

The map d_2 is given as follows. An FE-flag (F, E) is assigned $o(F, E) = 1$ is (F, E) is in the system of orientations, $o(F, E) = -1$ otherwise. Given an internal edge E adjacent to faces F, F', there are natural inclusions $\mathcal{A}_E \hookrightarrow \mathcal{A}_F, \mathcal{A}_{F'}$ inducing projections

$$\mathbb{R}^{\mathcal{A}_F} \to \mathbb{R}^{\mathcal{A}_E},\ \mathbb{R}^{\mathcal{A}_{F'}} \to \mathbb{R}^{\mathcal{A}_E}.$$

d_2 is defined so that projected onto the $\mathbb{R}^{\mathcal{A}_E}$ summand, it becomes

$$\mathbb{R}^{\mathcal{A}_F} \bigoplus \mathbb{R}^{\mathcal{A}'_F} \longrightarrow \mathbb{R}^{\mathcal{A}_E} \bigoplus \mathbb{R}^{\mathcal{A}_E} \xrightarrow{o(F,E)I \bigoplus o(F',E)I} \mathbb{R}^{\mathcal{A}_E}\ .$$

d_2 computes the difference between functions defined on faces F and F' along their common edge E

The map d_1 is given as follows. If v is an internal vertex, it is contained in internal edges $E_1, \ldots, E_k$ ordered counter-clockwise about v. Let $F_1, \ldots, F_k$ be the set of faces surrounding v so that E_i is adjacent to F_i and F_{i+1}. Define $o(E_i)$ to be

$+1$ if (F_{i+1}, E_i) is in the system of orientations, -1 otherwise. Let $\mathbb{R}^{\mathcal{A}_{E_i}} \to \mathbb{R}^{\{v\}}$ be the projections induced by inclusions. d_1 projected to $\mathbb{R}^{\{v\}}$ is

$$\bigoplus_{i=1}^{k} \mathbb{R}^{\mathcal{A}_{E_i}} \longrightarrow \bigoplus_{i=1}^{k} \mathbb{R}^{\{v\}} \longrightarrow \mathbb{R}^{\{v\}}$$

where the last map is given by

$$(f_1, \ldots, f_k) \mapsto \sum_{i=1}^{k} o(E_i) f_i.$$

Essentially, if we view the functions on $E_1, \ldots, E_k$ as the difference between the values of the adjacent faces, d_1 computes the monodromy around the vertex.

This sequence is easily seen to be exact. It is also well-defined and exact if $\mathbb{R}$ is replaced by $\mathbb{Z}$. Since there is no torsion in any term, we may take duals and obtain the exact sequence of Lemma 8.9.

8.2. **Exact sequence of secondary fan conditions.** In this subsection, we construct an exact sequence which will help us prove Proposition 8.10.

Proposition 8.17. *The following sequence is exact:*

$$0 \longrightarrow \bigoplus_{v \in I\mathcal{V}} \Lambda_v^{\perp} \longrightarrow \bigoplus_{E \in I\mathcal{E}} \Lambda_E^{\perp} \longrightarrow \bigoplus_{F \in \mathcal{F}} \Lambda_F^{\perp} \longrightarrow \Lambda_C^{\perp} \longrightarrow 0.$$

We break the exactness proof into several steps.

Lemma 8.18. *There is an exact sequence of linear subspaces of the terms in the above exact sequence*

$$0 \longrightarrow V \xrightarrow{i} \bigoplus_{F \in \mathcal{F}} V_F \xrightarrow{d_2} \bigoplus_{E \in I\mathcal{E}} V_E \xrightarrow{d_1} \bigoplus_{v \in I\mathcal{V}} V_v \longrightarrow 0.$$

Proof. The above sequence is clearly a chain complex, i is injective, $\ker(d_2) = \operatorname{im}(i)$, and d_1 is surjective. Therefore,

$$\dim V = \sum_{F \in \mathcal{F}} V_F - \sum_{E \in I\mathcal{E}} V_E + \sum_{v \in I\mathcal{V}} V_v + \dim\left(\frac{\ker(d_1)}{\operatorname{im}(d_2)}\right)$$

Comparing this formula with the equality in Lemma 4.4, we see that $\ker(d_1) = \operatorname{im}(d_2)$. $\square$

Lemma 8.19.

$$0 \longrightarrow \bigoplus_{v \in I\mathcal{V}} V_v^{\perp} \longrightarrow \bigoplus_{E \in I\mathcal{E}} V_E^{\perp} \longrightarrow \bigoplus_{F \in \mathcal{F}} V_F^{\perp} \longrightarrow V^{\perp} \longrightarrow 0$$

is an exact subsequence of the dual sequence

$$0 \longrightarrow \bigoplus_{v \in I\mathcal{V}} (\mathbb{R}^{\{v\}})^{\vee} \longrightarrow \bigoplus_{E \in I\mathcal{E}} (\mathbb{R}^{\mathcal{A}_E})^{\vee} \longrightarrow \bigoplus_{F \in \mathcal{F}} (\mathbb{R}^{\mathcal{A}_F})^{\vee} \longrightarrow (\mathbb{R}^{\mathcal{A}})^{\vee} \longrightarrow 0.$$

Proof. Given an exact sequence of linear subspaces

$$0 \longrightarrow W_0 \longrightarrow W_1 \longrightarrow \cdots \longrightarrow W_n \longrightarrow 0$$

contained in an exact sequence of vector spaces

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

we may dualize to get an exact sequence

$$0 \longrightarrow V_n^\vee \longrightarrow V_{n-1}^\vee \longrightarrow \cdots \longrightarrow V_0^\vee \longrightarrow 0$$

surjecting onto

$$0 \longrightarrow W_n^\vee \longrightarrow W_{n-1}^\vee \longrightarrow \cdots \longrightarrow W_0^\vee \longrightarrow 0\,.$$

The kernel in each degree is $W_i^\perp$. By examining the induced long exact sequence of

$$0 \to W_\bullet^\perp \to V_\bullet^\vee \to W_\bullet^\vee \to 0$$

we see that $W_\bullet^\perp$'s is exact. □

Intersecting the exact sequence of this lemma with that of Lemma 8.9 we get the exact sequence of Proposition 8.17.

8.3. **Exact sequence of point conditions.** Now, we prove Proposition 8.10 by showing the exactness of the following sequence:

$$0 \longrightarrow \bigoplus_{v \in I\mathcal{V}} \Lambda_v^\perp + \Lambda'_{PC,v} \xrightarrow{d_0} \bigoplus_{E \in I\mathcal{E}} \Lambda_E^\perp + \Lambda'_{PC,E}$$

$$\xrightarrow{d_1} \bigoplus_{F \in \mathcal{F}} \Lambda_F^\perp + \Lambda'_{PC,F} \xrightarrow{d_2} \Lambda_C^\perp + \Lambda'_{PC,\gamma} \longrightarrow 0$$

We will make use of the following observation: for the lattice index of $\Lambda_C^\perp + \Lambda'_{PC,\gamma}$ in $(\mathbb{Z}^{\mathcal{A}})^\vee$ to be non-zero, we must have, $\Lambda_C^\perp \cap \Lambda_{PC,\gamma} = 0$ for dimensional reasons. Likewise,

$$\Lambda_E^\perp \cap \Lambda'_{PC,E} = \Lambda_v^\perp \cap \Lambda'_{PC,v} = 0.$$

Therefore, the sum is direct in the first, second, and fourth terms of the sequence.

Proof. Let us note that

$$0 \longrightarrow \bigoplus_{v \in I\mathcal{V}} \Lambda'_{\mathrm{PC},v} \longrightarrow \bigoplus_{E \in I\mathcal{E}} \Lambda'_{PC,E} \longrightarrow \bigoplus_{F \in \mathcal{F}} \Lambda'_{PC,F}$$

is exact. It is clear that $\mathrm{im}(d_1) = \ker(d_2)$. It suffices to show $\mathrm{im}(d_2) \subseteq \Lambda_C^\perp + \Lambda'_{\mathrm{PC},\gamma}$ since d_2 is surjective because every point condition comes from an edge in the dual subdivision. Let us show that for any $v \in \mathcal{A}$, a vertex of the regular subdivision then $\phi_v \in \Lambda_C^\perp + \Lambda'_{\mathrm{PC}}$. A fortiori we will show that ϕ_v is in the lattice spanned by Λ'_{PC} together with all vectors of the form $\phi_{v_0} - \phi_{v_1} + \phi_{v_2} - \phi_{v_3}$ where v_0, v_1, v_2, v_3 are vertices of a parallelogram ordered counterclockwise. This lattice we call the *parallelogram-PC lattice*. It is clear that this lattice is contained in $\Lambda_C^\perp + \Lambda'_{PC,\gamma}$.

By Lemma 4.4, the number of point conditions is $V - 1 - \#\square$ where V and $\#\square$ are the numbers of vertices and parallelograms, respectively. Therefore, the number of trees in the point-condition forest is $\#\square + 1$. Notice that $\phi_v \in \Lambda'_{PC} \subseteq \Lambda_C^\perp + \Lambda'_{PC}$ for any vertex v in the same tree as the distinguished vertex (I, J).

If $\#\square = 0$, the point condition forest has only one tree and contains all vertices of the regular subdivision, so $\Lambda'_{PC,\gamma}$ contains ϕ_v for all vertices of the regular subdivision and we are done. We now induct on the number of parallelograms. We need the following simple graph theoretical lemma.

Lemma 8.20. *Consider n non-intersecting parallelograms in the plane and $n+1$ trees in the plane (some of which may be singletons) such that the union of the trees contains all vertices of the parallelograms and do not intersect the interiors of the parallelograms. Then there is a parallelogram that has two non-diagonal vertices belonging to the same tree.*

Proof. The proof is by induction. If $n = 1$, there are two trees. If no tree contains non-diagonal vertices, then each tree must contain a pair of diagonal vertices. But this is impossible since a path between a diagonal vertices of one parallelogram will separate the other vertices of that parallelogram.

Suppose the lemma is true for $n-1$. Consider the case of n parallelograms. Suppose no tree contains two non-diagonal vertices of the same parallelogram. Let P be some parallelogram. By the $n = 1$ case, at least one pair of diagonal vertices of P do not belong to the same tree. Erase P and draw an edge connecting the diagonal vertices of P that belong to different trees, giving a new forest with $n-1$ trees. This edge belongs to a tree T of the new forest. By induction, one tree of the forest must contain a pair of non-diagonal vertices of some parallelogram P'. By assumption, this tree must be T. Similarly, the other pair of diagonal vertices of P either (a) belong to the same tree which we call T' or (b) they belong to different trees that meet in non-diagonal vertices of a parallelogram which we'll call P''. Now erase P' an draw an edge between the vertices of P' that belong to T. We now have a circuit containing a pair of diagonal vertices of P which separates other vertices of P and contradicts both (a) and (b). $\square$

For $\#\square > 0$, one tree touches two non-diagonal vertices of some parallelogram, say v_0, v_1. If v_2, v_3 are the other vertices of that parallelogram, then $\Lambda_C^{\perp}$ contains

$$\phi_{v_0} - \phi_{v_1} + \phi_{v_2} - \phi_{v_3}$$

while Λ'_{PC} contains

$$\phi_{v_0} - \phi_{v_1}.$$

Therefore, $\Lambda_C^{\perp} + \Lambda'_{PC,\gamma}$ contains $\phi_{v_2} - \phi_{v_3}$. It follows that $\Lambda_C^{\perp} + \Lambda'_{PC,\gamma}$ contains the parallelogram-PC lattice of the graph obtained by subdividing the parallelogram along the diagonal containing v_0 and adding a point condition between v_2 and v_3. If this point condition tree contains a circuit, then the intersection multiplicity we were trying to compute is 0, so we may suppose that there are no circuits. By induction on the number of parallelograms, $\phi_v \in \Lambda_C^{\perp} + \Lambda'_{PC,\gamma}$. It follows that our sequence is exact. $\square$

8.4. **Lattice Index Computations.** Now, we compute the lattice index of $\Lambda_?^{\perp} + \Lambda'_{PC,?}$ in $(\mathbb{Z}^{\mathcal{A}_?})^{\vee}$ for $? = F, E, v$.

Let F be a face of the nodal effective regular subdivision so that F is either a triangle or a parallelogram. We produce $\mathbb{Z}$-basis for $\Lambda_F^{\perp}$. Subdivide F into little triangles of area $\frac{1}{2}$. Take a little triangle that has a vertex v_0 of F as a vertex. Let b, c be the other vertices of the little triangle. Note that $|(b-v_0)\times(c-v_0)| = 1$. Then $\{b-v_0, c-v_0\}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Therefore any lattice point in F may be written as

$$v = v_0 + k(b-v_0) + l(c-v_0) = (-k-l+1)v_0 + kb + lc.$$

A basis for L_F is

$$\{\phi_v - k\phi_b - l\phi_c + (k+l-1)\phi_{v_0} | v = v_0 + k(b-v_0) + l(c-v_0) \in F \setminus \{v_0, b, c\}\}$$

This is a $\mathbb{Z}$-basis since every element has a ϕ_v with coefficient 1 while that ϕ_v is present in no other element.

We now prove lemma 8.11

Proof. Let v_0, v_1, v_2 be the vertices of the triangle. We will compute the lattice index in the special case where v_1 and v_2 are distinct from b and c. The other cases are similar. We write

$$v_1 = kb + lc - (k+l-1)v_0,\ v_2 = mb + nc - (m+n-1)v_0.$$

The point condition lattice is generated by $\{\phi_{v_1} - \phi_{v_0}, \phi_{v_2} - \phi_{v_0}, \phi_{v_0}\}$ or equivalently by $\{\phi_{v_0}, \phi_{v_1}, \phi_{v_2}\}$. We compute the lattice index as a determinant. Note that because for $v \in F \setminus \{v_0, v_1, v_2, b, c\}$, ϕ_v occurs in only one generator of the lattice and with coefficient 1. Therefore, when computing the determinant, we may eliminate the row and column corresponding to v, changing the determinant by, at most, a sign. Therefore, the lattice index is equal to

$$\begin{vmatrix} -k-l+1 & 1 & 0 & k & l \\ -m-n+1 & 0 & 1 & m & n \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} k & l \\ m & n \end{vmatrix}$$

where in the first matrix, the rows correspond to v_1, v_2, v_0, b, c. But this determinant is exactly twice the area of the triangle. □

We now prove Lemma 8.12

Proof. Let us denote the vertices of the parallelogram encountered counterclockwise as v_0, v_1, v_3, v_2. Then $\Lambda_F^\perp$ contains the vector $\phi_{v_0} - \phi_{v_1} + \phi_{v_2} - \phi_{v_3}$ and we may replace the basis element of $\Lambda_F^\perp$ corresponding to v_3 by this vector without changing the lattice index. The point conditions are placed on two non-parallel edges of the parallelogram. Therefore, the lattice index is equal to

$$\begin{vmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ -k-l+1 & 1 & 0 & 0 & k & l \\ -m-n+1 & 0 & 1 & 0 & m & n \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} k & l \\ m & n \end{vmatrix}$$

which is the area of the parallelogram. □

We now prove Lemma 8.13

Proof. Label the vertices of the edge by $\{v_0, v_1, \ldots, v_e\}$. Then the vectors

$$\phi_{v_i} - i\phi_{v_1} + (i-1)\phi_{v_0}, \quad i = 2, \ldots, e$$

form a basis for L_E. The lattice index is

$$\begin{vmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 2 & -3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ e-1 & e & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 \end{vmatrix} = e$$

□

9. Geometric Interpretation

We would like to give a geometric interpretation of the invariants we define.

Let $\mathbb{P}^{|\mathcal{A}|-1}$ denote the space of hypersurfaces in $X_{\mathcal{A}}$. In other words, a point $[a_{ij}] \in \mathbb{P}^{|\mathcal{A}|-1}$ defines a hypersurface $\sum_{(i,j)\in\mathcal{A}} a_{ij}x^iy^j$. The $(\mathbb{C}^*)^2$-action on $X_{\mathcal{A}}$ induces an action of $\mathbb{P}^{|\mathcal{A}|-1}$. There are two important quotients by this action, the Hilbert quotient $(\mathbb{P}^{|\mathcal{A}|-1})///(\mathbb{C}^*)^2$ and the Chow quotient $(\mathbb{P}^{\mathcal{A}|-1})//(\mathbb{C}^*)^2$. See [8] for background. Elements of the big open torus in each toric variety correspond to hypersurfaces in $X_{\mathcal{A}}$ defined up to $(\mathbb{C}^*)^2$-action. The secondary fan is the fan associated with the toric variety $\mathbb{P}^{|\mathcal{A}|-1}//(\mathbb{C}^*)^2$ [6]. The Hilbert quotient has a universal flat family $\mathcal{U} \subset (\mathbb{P}^{|\mathcal{A}|-1}///(\mathbb{C}^*)^2) \times \mathbb{P}^{|\mathcal{A}|-1}$ which should be thought of as a family of toric surfaces over the moduli space $(\mathbb{P}^{|\mathcal{A}|-1})///(\mathbb{C}^*)^2$. They contain degenerations of $X_{\mathcal{A}}$ that catch components of the hypersurface that break off in limits of the $(\mathbb{C}^*)^2$-action. There is a natural fundamental cycle map

$$fc : (\mathbb{P}^{|\mathcal{A}|-1})///(\mathbb{C}^*)^2 \to (\mathbb{P}^{|\mathcal{A}|-1})//(\mathbb{C}^*)^2.$$

and a composition

$$e : \mathcal{U} \hookrightarrow (\mathbb{P}^{|\mathcal{A}|-1}///(\mathbb{C}^*)^2) \times \mathbb{P}^{|\mathcal{A}|-1} \to \mathbb{P}^{|\mathcal{A}|-1}///(\mathbb{C}^*)^2$$

We have the following maps:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "e"] \arrow[d, "p"] & \mathbb{P}^{|\mathcal{A}|-1} \\ (\mathbb{P}^{|\mathcal{A}|-1})///(\mathbb{C}^*)^2 \arrow[d, "fc"] & \\ (\mathbb{P}^{|\mathcal{A}-1})//(\mathbb{C}^*)^2 & \end{tikzcd}$$

A point condition hyperplane $H_{(\overline{x},\overline{y})} \subset \mathbb{P}^{|\mathcal{A}|-1}$ is defined by

$$\sum a_{ij}\overline{x}^i\overline{y}^j$$

for $(\overline{x},\overline{y}) \in (\mathbb{C}^*)^2$. By the connection between tropical fans and Minkowski weights, an m-dimensional tropical fan on Σ corresponds to a Chow cohomology class $c \in A^m((\mathbb{P}^{|\mathcal{A}|-1})//(\mathbb{C}^*)^2)$. By the correspondence between classical and tropical intersection theory [10], our invariants are the intersection product

$$\deg((fc\circ p)^*c \cup e^*(H_{(\overline{x}_1,\overline{y}_1)} \cup \cdots \cup H_{(\overline{x}_m,\overline{y}_m)}))$$

provided such intersections take place in the open torus in $\mathcal{U}$. We are therefore looking at families of hypersurfaces in a toric surface $X_{\mathcal{A}}$ which may degenerate. The class c forces the toric surface to degenerate while the point condition hyperplanes force the hypersurface to pass through certain points $(\overline{x}_k, \overline{y}_k)$. This idea is very similar to the theory of relative Gromov-Witten theory [11, 12] where we must allow our target space to degenerate and so create a moduli of targets. The moduli of targets have cohomology classes which can be pulled back and used to enrich Gromov-Witten invariants. See [9] for details.

The Hilbert quotient of the Severi variety $V_\delta///(\mathbb{C}^*)^2$ embeds in the Hilbert quotient $(\mathbb{P}^{|\mathcal{A}|-1})///(\mathbb{C}^*)^2$. We suspect that even though the ambient Hilbert quotient is non-smooth, $V_\delta///(\mathbb{C}^*)^2$ possesses a sort of Poincare-dual as a rational operational Chow cohomology class. We believe but are unable to prove that the fan of Example 6.6 are the values of this operational class on nodal effective cones.

10. A Counterexample

In this section, we show that the tropicalization of the Severi variety is not supported on the secondary fan. Given a closed lattice polytope $P \subset \mathbb{Z}^2$, let $\mathcal{A} = P \cap \mathbb{Z}^2$. Let $\mathbb{P}^{|\mathcal{A}|-1}$ be the space of curves in the toric surface S associated to $\mathcal{A}$. Let us assume that P is chosen so that S is smooth. Let $V_\delta \subset \mathbb{P}^{|\mathcal{A}|-1}$ be the Severi variety, the closure of the locus of curves with exactly δ nodes and no other singular points. The dimension of V_δ is $\delta = |\mathcal{A}| - \delta - 1$.

The tropicalization of the Severi variety, $T = \mathrm{Trop}(V_\delta) \subset T\mathbb{P}^{|\mathcal{A}|-1}$ is the image of the Severi variety under the order map. We will view the tropicalization as lying in $\mathbb{R}^{\mathcal{A}}$. Because V_1 corresponds to the $\mathcal{A}$-discriminant, $\mathrm{Trop}(V_1)$ is supported on the secondary fan by the Prime Factorization Theorem of [6].

We identify a cone in $T = \mathrm{Trop}(V_2)$ not supported on the secondary fan. Our method will be to find a curve which under the order map corresponds to a point in the relative interior of a top-dimensional cone in T but not contained in a cone of the same dimension in the secondary fan.

Let P be the square with vertices $(0,0), (2,0), (0,2), (2,2)$. The toric surface is $\mathbb{P}^1 \times \mathbb{P}^1$ and points in $\mathbb{P}^{|\mathcal{A}|-1}$ correspond to curves of degree $(2,2)$. We will identify a curve in $V_2(S)$ (defined over the field of Puiseux series, $\mathbb{K} = C\{\{t\}\}$). This curve will correspond to a point in T that is the relative interior of a top, that is 7-dimensional, cone but does not lie in any 7-dimensional cone of the secondary fan of P.

Let a, b, c, d be rational numbers satisfying

(1) $0 < a < c$
(2) $b > 2a$
(3) $d > 2c$

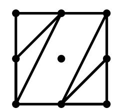
Consider the polynomial

$$\begin{aligned} f(x,y) &= (1+x)(1+t^d x + t^a y + t^c xy + t^b y^2 + xy^2) \\ &= 1 + (t^0 + t^d)x + t^d x^2 + t^a y + (t^a + t^c)xy \\ &\quad + t^c x^2 y + t^b y^2 + (t^0 + t^b)xy^2 + x^2y^2 \end{aligned}$$

The corresponding hypersurface is a curve with two nodes. Under the order map, this gives a point in the relative interior of a 7-dimensional cone in the tropicalization of the Severi variety. This is because there are four degrees of freedom coming from varying a, b, c, d and three additional degrees of freedom from the following:

(1) Multiplying f be a constant in $\mathbb{K}$;
(2-3) Transforming $f(x, y)$ to $f(px, qy)$ for arbitrary $p, q \in \mathbb{K}$.

We claim that the tropicalization of f does not lie in the relative interior of any 7-dimensional cone of the secondary fan. The subdivision of P corresponding to f is pictured below.



Therefore, the only 7-dimensional cone in the secondary fan that could contain f is the one corresponding to this subdivision. It consists of curves

$$\sum_{(i,j)\in\mathcal{A}} a_{ij}x^i y^j$$

where

(1) $\mathrm{ord}(a_{12}) = \mathrm{ord}(a_{00}) + 2(\mathrm{ord}(a_{11}) - \mathrm{ord}(a_{00})) - (\mathrm{ord}(a_{10}) - \mathrm{ord}(a_{00}))$
(2) $\mathrm{ord}(a_{22}) = \mathrm{ord}(a_{00}) + (\mathrm{ord}(a_{12}) - \mathrm{ord}(a_{00})) + (\mathrm{ord}(a_{10}) - \mathrm{ord}(a_{00}))$

together with some inequalities.

But in f, $\mathrm{ord}(a_{11}) = a > 0$ while (1) above mandates that $\mathrm{ord}(a_{11}) = 0$.

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Department of Mathematics, University of Texas
E-mail address: eekatz@math.utexas.edu